Absence of the link between self-organized criticality and deterministic fixed energy sandpiles

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Self-organized criticality (SOC) observed in an abelian sandpile model (ASM) has been believed to be related to the absorbing phase transitions (APT) exhibited by a deterministic fixed energy sandpile (DFES). We critically investigate the link between the SOC and the APT exhibited by the DFES. In contrast to the widespread belief, a phase transition in the DFES is shown not to be defined uniquely but to depend on initial conditions because of the non-ergodicity of the DFES. Furthermore, we show that a phase transition in the spreading dynamics of the DFES is intimately related to the percolation rather than the avalanche dynamics of the ASM. These results illustrate that the SOC exhibited by the ASM has nothing to do with phase transitions in the DFES. We discuss the implication of our result to stochastic models.

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The concept of self-organized criticality (SOC) coined by Bak, Tang, and Wiesenfeld (BTW) [1] has been invoked to describe the ubiquity of criticality in nature [2, 3]. Sandpile models with either deterministic [1, 4] or stochastic [5] toppling rule have played the role of Ising model in understanding SOC theoretically [6, 7]. Although sandpile models were devised as prototypes with the spontaneous emergence of criticality without fine-tuning of any parameters, which is the characteristics of SOC [1], the similarity of SOC to standard (nonequilibrium) phase transitions had been discussed immediately after the emergence of the concept [8, 9]. The connection between SOC and standard phase transitions, especially absorbing phase transitions (APTs) [10, 11], has been materialized by defining a 'fixed energy' ensemble of the sandpile model, or a fixed energy sandpile (FES) [12–15]. The idea of the FES has elicited fruitful conclusions at least for stochastic sandpile models; the FES version of the stochastic sandpile model [5] is argued to form a universality class together with other APT models [13], which, in turn, has motivated to pave "paths to self-organized criticality" from conventional phase transitions [16, 17].

Despite the triumph of the idea of the FES, skepticism against the SOC-FES link has been unabated [6, 17, 18]. In comparison to the achievement in the analytic study of deterministic sandpile models, also known as abelian sandpile models (ASMs) [4], rigorous analysis in favor of the link between SOC and the FES still lacks. In this context, it is not surprising that the authors of Ref. [12] also worried about the possibility that the FES may explore transient rather than recurrent (SOC) configurations. Thus, a first step to remove such skepticism would be to cement the link between the SOC in the ASM and its FES version or deterministic FES (DFES). The purpose of this work is to investigate if the above-mentioned program can be accomplished.

However, this program does not seem fully successful. As an easy counterexample, it is clear that the avalanche dynamics of the one-dimensional BTW automata [19] is completely different from dynamics at transition point of its FES version [20]. Although the ASM and the DFES in the one-dimensional chain do not share criticality, the SOC critical density is identical to the transition point of the DFES [19, 20]. Accordingly, it seems still valid to anticipate that the mean energy in an ASM hovers around the phase transition point ζ_c of the corresponding DFES by dissipation (when activity is too strong) and driving (when activity is absent) and that in the thermodynamic limit the mean energy ζ_S of the ASM at criticality is identical to ζ_c. Meanwhile, the ASM on a one-dimensional chain is rather pathological in the sense that the probability of observing any finite avalanche in the thermodynamic limit is zero [19]. Hence, one might expect that except such pathological cases the program to associate the SOC with the DFES may turn out to be successful.

Recently, however, even this expectation has been challenged [18]. By measuring ζ_c in the thermodynamic limit which is preceded by the infinite time limit, Fey, Levine, and Wilson [18] argued that ζ_c is different from ζ_S for many different models. Because phase transitions in nonequilibrium systems are usually studied in the thermodynamic limit followed by not preceded by the infinite time limit, the method employed in Ref. [18] might be considered inconsistent with the philosophy of the FES. Then can we restore the link between the SOC and the DFES if appropriate order of limits are taken?

To answer the above question, let us begin with defining the DFES. In the DFES on a graph with V vertices, each vertex starts with nonnegative integer energy z_i ($i = 1, \ldots, V$). A configuration change is governed by a $V \times V$ integer matrix F which has properties such that $F_{ij} \leq 0$ for $i \neq j$, $\sum_j F_{ij} = 0$ (energy conservation), and for any pair i, j of vertices there is a sequence, say a connecting sequence, of vertices with finite length

$l+2$ ($k_0 = i, k_1, \ldots, k_{l+1} = j$) such that $F_{k_m k_{m+1}} < 0$ for all $0 \le m \le l$. Except the energy conservation, these properties are same as in the ASM [6]. For convenience, $z_i^c \equiv F_{ii} - 1$ is introduced and called the critical energy (at vertex i). We refer to a vertex i with energy higher than the critical energy as an unstable vertex and it can topple, resulting in the energy redistribution such that $z_j \to z_j - F_{ij}$ for all j. It is clear that the total energy of a configuration C, $E(C) = \sum_i z_i$, is conserved during toppling events.

By a sequence of toppling events, a system may evolve to a configuration without an unstable vertex, which will be called an absorbing state. For convenience, we introduce a function of configurations $A(C)$ which takes 0 if the system starting from C will arrive at an absorbing state after a finite number of toppling events and 1 otherwise. Due to the abelian property of toppling events [4], this function does not depend on the way how toppling events are ordered and, accordingly, is well-defined for finite V.

For the DFES, there are two bounds Z_M and Z_m such that

$$Z_M = \min\{E|\forall_C \text{ with } E(C) > E, A(C) = 1\}, \quad (1)$$
$$Z_m = \max\{E|\forall_C \text{ with } E(C) < E, A(C) = 0\}. \quad (2)$$

It is easy to get $Z_M = \sum_i z_i^c$. The existence of Z_m is obvious, but to find Z_m for arbitrary F does not look simple. Fortunately, if $\sum_i F_{ij} = 0$, that is, if there is no greedy vertex [6], we can show that $Z_m = -\sum_{i \neq j} F_{ij}/2 = \sum_i F_{ii}/2$. To this end, we notice that $A(C) = 1$ under the simultaneous parallel update (SPU) implies that the steady state is a limit cycle with a certain period T [21, 22]. In one period, every vertex should topple exactly same number of times, say W, because there is no greedy vertex. Accordingly, the total energy flow between vertices is $2WZ_m$. Hence the total energy should not be smaller than Z_m.

To complete the proof, we will construct a configuration C_m with $A(C_m) = 1$ and $E(C_m) = Z_m$ as follows. Let P be a permutation of $1, \ldots, V$. At first, the vertex $P(1) = i$ is given energy F_{ii}. Energy at other vertices are assigned by the following iteration; if the vertex $P(k)$ has been assigned energy, the energy of vertex $P(k+1)$ becomes $-\sum_j' F_{P(k+1),j}$, where the sum is over all vertices except $P(1), \ldots, P(k+1)$. The finial configuration after all vertices are exhausted is C_m. It is obvious that $E(C_m) = Z_m$. If an ASM is defined by the toppling matrix $\Delta_{ij} = F_{ij} + \delta_{ij} L_i$, where $L_i = 1$ if the energy at vertex i of C_m is F_{ii} and 0 otherwise, C_m should be a recurrent state of this ASM. This is because by construction it should successfully go through the burning test [4]. Note that the burning test to find a recurrent state is applicable when there is no greedy vertex. If we add L_i to all vertices to C_m and perform the toppling dynamics according to Δ (with dissipation), the final state should be C_m after all vertices topple exactly once because adding energy of amount L_i to all vertices of a recurrent state of this ASM is an identity operation [4]. That is, by the DFES toppling rule, C_m should come back to C_m if the toppling order is taken exactly as the corresponding ASM, which implies $A(C_m) = 1$. Note that the minimum energy among recurrent states of this ASM is exactly Z_m [6]. Hence above consideration also proves that $A(C)$ of a recurrent state C of a certain ASM constructed by making some vertices dissipative should be 1.

Z_M is in many cases larger than Z_m, but the order relation is reversed for the DFES on a tree. By a tree is meant a system with $|F_{ij}| = |F_{ji}| \le 1$ ($i \neq j$) and a unique connecting sequence between any two different vertices. In a tree, Z_M is $V-2$ and Z_m is $V-1$. Hence for any configuration on a tree with $E(C) \ge Z_m = V-1$, $A(C) = 1$.

Since $A(C) = 0$ for any C with $E(C) < Z_m$ and $A(C) = 1$ for any C with $E(C) > Z_M$, one might expect that "the phase transition point" ζ_c should lie between ζ_m and ζ_M, where ζ_m (ζ_M) is the limiting value of Z_m/V (Z_M/V) in the thermodynamics limit $V \to \infty$. Now we will investigate if this claim is valid.

In the literature, the similarity between this phase transition and the usual APT has been emphasized [13]. However, the DFES as an APT model has a certain property which is not possessed by usual APT models such as the contact process (CP) [23]. In the CP, a finite system should fall into an absorbing (particle vacuum) state with probability 1 unless spontaneous death is prohibited. On the other hand, some finite systems in DFES cannot fall into an absorbing state. Since the phase transition involves the thermodynamic limit as well as the infinite time limit, the order of these two limits is crucial. In the CP, if infinite time limit precedes the thermodynamic limit, there is no nontrivial phase transition (everywhere inactive). However, if these two limits are taken in the opposite order, there is a nontrivial phase transition which is the main concern in the literature. In the DFES, however, nontrivial phase transition can be defined even if we take the infinite time limit first [18]. Our discussion about the APT exhibited by the DFES starts from this "unconventional" phase transition.

To study an APT, it is also necessary to define the (mean) density of unstable vertices at stationarity, say $\rho(C)$, where C is the initial configuration. If $A(C) = 0$, $\rho(C)$ is unambiguously 0. However, $\rho(C)$ depends on how the toppling events are ordered if $A(C) = 1$ [24]. For example, the devil's staircase [21] is observable when the SPU is employed, but it should disappear if a random sequential update rule is employed because the steady state is not a limit cycle any more. In any case, it is trivially true that $A(C) \ge \rho(C)$ for all C. To simplify the discussion, we will exclusively assume the SPU.

Let $P_0(C; \zeta, V)$ be the probability that an initial con-

figuration of the DFES with V vertices is C. Here, ζ is the mean energy per vertex satisfying $V\zeta = \sum_C E(C)P_0(C;\zeta,V)$. Let us introduce

$$\phi_1(\zeta;V) = \sum_C A(C)P_0(C;\zeta,V), \tag{3a}$$

$$\phi_2(\zeta;V) = \sum_C \rho(C)P_0(C;\zeta,V), \tag{3b}$$

and define "transition points" ($\ell = 1$ or 2)

$$\zeta_c^{(\ell)} = \inf_\zeta \left\{ \zeta \,\middle|\, \lim_{V\to\infty} \phi_\ell(\zeta;V) \neq 0 \right\}. \tag{4}$$

Since $\phi_1 \geq \phi_2$, $\zeta_c^{(2)}$ cannot be smaller than $\zeta_c^{(1)}$. By definition, $\zeta_m \leq \zeta_c^{(1)} \leq \zeta_M$. Although the equality $\zeta_c^{(1)} = \zeta_c^{(2)}$ seems plausible for any DFES, we could not find a general proof. We can prove this equality only for the DFES defined on a tree because the period of a limit cycle on a tree is either 1 or 2 [25], which implies either $A(C) = 2\rho(C)$ or $A(C) = \rho(C)$ (that is, $\phi_2 \leq \phi_1 \leq 2\phi_2$). Hence, $\zeta_c^{(2)} = \zeta_c^{(1)} = 1$ is satisfied on every tree regardless of the initial conditions. Note that the method used in Ref. [18] to find the transition point is equivalent to using ϕ_1 with Poisson distributed initial condition.

In fact, the definition (4) makes it impossible to define "the" transition point. $\zeta_c^{(1)}$ should depend on P_0. To explain why, let us divide the whole configurations into two classes S^I and S^A such that if $C \in S^I$ ($C \in S^A$), $A(C) = 0$ (1). Due to the non-ergodicity of the DFES [21], S^I and S^A are disjoint and should exhaust all possible configurations. Let $\zeta_A(\zeta;V) = \sum_{C\in S^A} P_0(C;\zeta,V)E(C)/V \geq \phi_1 Z_m/V$. If one can choose P_0 such that in the thermodynamic limit $\zeta_A \to f(\zeta)\theta(\zeta-\zeta_0)$, where $\zeta_m \leq \zeta_0 \leq \zeta_M$, θ is Heaviside step function, and $f(\zeta)$ which should be not larger than ζ is a certain function, this model shows a (possibly discontinuous) transition at $\zeta_c^{(1)} = \zeta_0$. Only in case $\zeta_m = \zeta_M$ as in the DFES on a tree, the transition point (4) of the FES is unambiguously defined. Even worse, certain $P_0(C;\zeta,V)$ can trigger reentrance behavior in the sense that the set $\{\zeta|\phi_1(\zeta;\infty) = 0\}$ is not connected. In this context, the disagreement between ζ_c (actually $\zeta_c^{(1)}$) and ζ_S observed in Ref. [18] is the generic feature of the DFES.

To illustrate, we will investigate the DFES on a bracelet graph with periodic boundary conditions defined by $F_{ij} = 4\delta_{ij} - 2\delta_{|i-j|,1} - 2\delta_{|i-j|,V-1}$. From the general consideration given above, $\zeta_m = 2$ and $\zeta_M = 3$. Hence, $\zeta_c^{(1)}$ should be located between 2 and 3. Now assume that the initial energy distribution at each vertex is drawn from

$$\begin{aligned} p(2k;\zeta) &= (1-p_o(\zeta))\left(\frac{\zeta}{2}\right)^k \frac{e^{-\zeta/2}}{k!}, \\ p(2k+1;\zeta) &= p_o(\zeta)\left(\frac{\zeta-1}{2}\right)^k \frac{e^{-(\zeta-1)/2}}{k!}, \end{aligned} \tag{5}$$

where k is a non-negative integer and $p_o(\zeta)$ is a certain function of ζ in the range $0 \leq p_o(\zeta) \leq 1$. Due to the parity conservation at each vertex [18] and its resemblance to the DFES on a chain, one can easily find that if ζ is smaller (larger) than $2 + p_o(\zeta)$, $\phi_1(\zeta;V)$ goes to zero (nonzero) in the thermodynamic limit. That is, phase transitions occur at points where $2 + p_o(\zeta) - \zeta$ changes sign. For example, let $p_o(\zeta) = [4\zeta + 1] \pmod 2$, where $[x]$ means the largest integer not greater than x and the modulo 2 operation restricts the possible values to either 0 or 1. Note that $2 + p_o(x) - x$ changes its sign at $\frac{9}{4}, \frac{5}{2}, \frac{11}{4}$. Hence, there is reentrance behavior in the phase diagram (absorbing, active, absorbing, then active phases as ζ increases). For other initial condition, one should use $p_o(\zeta) = \sum_{k=0}^\infty p(2k+1;\zeta)$ for the criterion of phase transition.

As argued in the beginning, the definition of transition points Eq. (4) does not comply with the spirit of the usual APT. A proper definition consistent with the usual APT is

$$\zeta_c^{(3)} = \inf_\zeta \{\zeta | \lim_{t\to\infty} \phi_3(\zeta,t) \neq 0\}, \tag{6}$$

$$\phi_3(\zeta,t) = \lim_{V\to\infty} \sum_C n(C_t)P_0(C;\zeta,V), \tag{7}$$

where C_t is the configuration at time t from the initial configuration C, and $n(C_t)$ is the density of unstable vertices averaged over time up to t [$\rho(C) = \lim_{t\to\infty} n(C_t)$]. But note that $\zeta_c^{(3)}$ cannot be larger than $\zeta_c^{(2)}$, so it is clear that $\zeta_c^{(3)}$ cannot be defined uniquely, either.

Since $\zeta_m \leq \zeta_S \leq \zeta_M$ for a certain ASM constructed from the DFES with negligible dissipative vertices, it may be possible to find an initial condition which yields the criticality as in the ASM (fine-tuning of initial conditions). However, as we have seen, recurrent states are related to the active phase within our definition, so the criticality, if exists, has nothing to do with the recurrent states. If the ASM forms a universality class, the detailed structure of dissipative vertices should not yield a considerable difference. Hence, no DFES can exhibit the similar critical behavior as the ASM, if universality hypothesis is valid.

The difference between the ASM and the DFES is even more striking when we consider a spreading dynamics, or a microscopic APT (mAPT) [26]. The mAPT studies how an initially localized unstable vertices spreads through a background inactive region with infinite volume (the density of unstable sites is zero for all time, hence the name "microscopic"). The order parameter in this case is the (survival) probability that the activity persists indefinitely. For illustration, our discussion will be restricted to the two-dimensional BTW model [1].

First, we construct the background inactive region in such a way that each site is assigned energy 0, 1, 2, or 3 with the probability $(1-p_3)p_0$, $(1-p_3)p_1$, $(1-p_3)p_2$, or p_3,

respectively ($0 \le p_i \le 1$ and $p_0 + p_1 + p_2 = 1$). By varying p_i's, one can study the mAPT with the background density ranging from 0 to 3. After forming the background density, we choose a site randomly (this is not a mathematically rigorous statement, but the rigor can be attained with ease) and perform the DFES dynamics. Obviously, no unstable site appears with probability $1 - p_3$. If p_3 is larger than the site percolation threshold [27] and if the randomly chosen site is in the infinite cluster of sites with energy 3, the spreading dynamics never dies. Hence, the critical density ζ_{mc} is bounded by

$$\zeta_{mc} \le (1 - p^*)(p_1 + 2p_2) + 3p^*. \tag{8}$$

Hence by varying p_1 and p_2, one can have infinitely many transition points. In striking contrast to the previous discussion, the critical density can be smaller than $\zeta_m = 2$. For example, if $p_1 = p_2 = 0$, it is clear that $\zeta_{mc} = 3p^* \simeq 1.778\ 24 < 2$. Note that the numerical value of the site-percolation threshold is $p^* \simeq 0.592\ 746$ [27].

Up to now, we have shown that phase transitions occurring in the DFES without greedy vertices have nothing to do with the SOC displayed by the corresponding ASM and that the program introduced in the beginning cannot be fulfilled. This naturally raises the following question. Can we make a similar conclusion for the relation between stochastic sandpile models and its FES version, or stochastic FES (SFES)?

Clearly, one cannot apply the discussion in the above directly to stochastic sandpile models. Unlike the DFES, the mAPT of the SFES cannot be understood in the framework of the percolation. That is, even if there is a percolating cluster with low background energy density, it is very probable that this system will fall into an absorbing state. Or even if there is no percolating cluster with sufficiently large background density, the stochasticity can allow the activity to jump to another (finite) cluster and can survive indefinitely with nonzero probability. Besides, any finite system of the SFES, unless total energy is larger than Z_M, should fall into an absorbing state in the infinite time limit just as in the CP. In other words, a recurrent state of a finite system can evolve to an absorbing state of the corresponding SFES.

However, the recent observation by Karmakar *et al.* [28] suggests an indirect application of our discussion to the SFES. According to Karmakar *et al.* [28], a certain deterministic sandpile model belongs to the same universality class as the stochastic Manna model [5]. Since the critical density of a DFES cannot be uniquely defined as we have seen, a transition point of the FES version of the model in Ref. [28] is not defined uniquely. If the universality hypothesis in the sandpile models also implies the same universality class of the FES versions, the critical behavior of the SFES would have nothing to do with the stochastic sandpile model. Of course, this claim becomes meaningless, if the universality hypothesis is not applicable to the FES version or if the claim in Ref. [28] is wrong. However, the numerical study in Ref. [28] is rather convincing and the universality hypothesis seems valid [13]. In any case, we believe that it is still necessary to make the connection between the SFES and the stochastic sandpile models more firmly.

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